# ВЫЧИСЛИТЕЛЬНЫЕ И ЕСТЕСТВЕННО-НАУЧНЫЕ АСПЕКТЫ

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## ЧТО ЕСТЬ СТРОГОЕ ДОКАЗАТЕЛЬСТВО?

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## WHAT SHOULD A STRICT PROOF BE LIKE?

Everyone has ever come across mathematical proofs. Those who presently do without them, often recall elementary geometry course, given at school, like nightmare or at least something extremely detached from everyday life, for though the term "geometry" means "measuring land". Even those, who happened to do with so-called "higher mathematics", often have no enthusiasm for scopes and means of proofs they came across.

By academician A.N. Krylov, strictness of reasoning self-contained leads to extreme formalization that inevitably results in "triumph of science over good sense" [1]. Discussed below, are the ways of withstanding this trend.

It is unanimously approved that one would take a proof of a puzzling theorem much easier, if it were previously lectured. Moreover, although the pace of a lecture is imposed, and a textbook may be laid behind, students get at lectured proofs much better than at bookish ones. We shall start with the latter, and the former will be considered later.

It is quite evident that only written language enables consistent exposition and formalization of mathematical reasoning, i.e. mathematics itself. In what aspect may modern computers be helpful to expose and understand mathematical deductions? Could they provide new opportunities for getting at the principal ideas and could they foster new discoveries?

There are many examples, how new ways of notation stimulated original approaches and gave rise to new results in the long run. Among them are Dynkin diagrams and diagrams in homological algebra that were humorously called by their author S. McLane "abstract rubbish" first; this notion is vital in literature at present [2]. Convenience of form or setting forth usually proves to be just a tip of an iceberg of new idea, it implies. Shrödinger's attempt to find a new appearance for the results obtained by Heizenberg, yielded the equation that was laid in the base of quantum mechanics. This is the fact that "pioneer's works are always awkward" [3], as G. Littlewood has justly mentioned. Thus is emphasized, that scrupulous processing is usually required for "journal" formulations and proofs.

It is worth mentioning that the aforesaid awkwardness of an originally cogitating person is by no means connected with one's weakness in methodology. Very often, the author of an ingenious but quite "unreadable" paper brilliantly delivers the deductions of other authors. The matter is that the very core of research looks insight and on the outsight not quite alike. This results in some shift on the scale of values and difficulties of the character in question. This is the reason for a paper "to have a rest" before being published.

Coming back to computers, we are affected by the following: if new powerful tools for visualization such as enhanced graphics, hypertext systems, multitasking, animation, etc. are useful for improved exposition of basic ideas of proof, and

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if the principal revision of the traditional ways of setting forth mathematics can follow.

There is such an opportunity, we believe. To answer this question we come close to the problem what should a strict proof be like, since this perception is inseparable from the traditional methods of exposition and even bases upon them.

The common approach is as follows: there are basic non-definable concepts, i.e. axioms, and logical rules to derive new results from those already proved to be valid, and rules of creation of new definitions. The successive set of statements, either axioms, or rightful derivatives from the preceding members of the set, constitutes a proof, with the last member of the set being the thing of interest [4].

We would like to elaborate the foregoing: references to statements proved in other texts are admitted for use. By the way, axioms and rules are somewhat contemptuous, but this question goes out of our consideration. We suppose, a text that may be included in some rightful proof in accordance with the rules above is true, if starting points are true. By academician Krylov, "like a millstone, mathematics grinds everything it is injected with, and you will not get truth from false starting points even if you cover pages with formulas, like flour does not arise from goose foot" [1].

The ideal standard of formal text is briefly portrayed above. Indeed, are the requirements to proofs of real papers and monographs etc. quite similar? Can they be considered as the examples of ideal texts? Obviously, they never can. At least, two reasons for it are present.

First, even misprints, usually available, provide a motivation to plead the text non-ideal. This problem is not so negligible as it seems at the first sight. Of course, the misgrinds such as this one, do not violently disturb the sense, and they can easily be corrected. But if you do with the table of values of some quantity, and at a particular position '2' was printed instead of '3', and '9' was missed at all, you would have more serious problems to verify truth. Unfortunately, situation of this kind is especially typical for filling in a table by computer, whose printer is not dependable and sometimes goes wrong. If computer accidentally falls out of step, this may also be related to misprints. This was the concern about a misprint probable to occur during the prolonged calculations held by computer in order to verify the hypothesis of four colorations, that resisted against the efforts of mathematicians in numerous attempts to prove it for more than a century, that shattered the confidence in its truth [5].

Second, all the texts appeal to reader's intuition ("this is obvious...", "no doubt, that...", etc.) or to reader's ability to go through some particular steps of proof oneself ("this immediately follows....", "calculations lead to...", etc.). Sometimes logically consistent proof order may be changed, and a lemma, necessary for the proof of some theorem, is delivered at the end of a section, devoted to the theorem. There are special words to stimulate reader's intuition ("immediately", "in other words", etc.). All the foregoing examples are extracted from [6] monograph that is extremely accurate in sense of material exposition. What will one say about the other monographs! By the way, it is sufficient to open any course of mathematics to verify the actuality of the things in discussion. Such concessions to evidence to the detriment of mathematical strictness are due to author's inspiration (N. Burbaci, in this case) to adapt the written text closer to its oral analogue in order to alleviate the apprehension by the reader. Not surprisingly, proofs presented in papers and even monographs, are usually referred to as not "formal" but "strict".

What is the reason for such a mismatch of formal and real texts? The first, "official" one is the economizing of paper. The second, the foremost, is the compromise to the human reasoning that is not formal. Nevertheless, it is able to draw even a top ace mathematician to blunders. For example, [7] demonstrates what a nasty trick may geometrical intuition play if applied to the theory of functions of real variables. Trying to discern the affordable degree of compromise, we come in touch with the question the present article is entitled with.

We believe, that strict proof is "something", usually a text, by means of what the reader is able to produce the formal proof, if desirable, and the ways to do it are directly explicit. The point of view that demands the entire formalization in order to fulfil computer check that removes the idea of an original interpretation, is by no means fit for this purpose, just owing to misprints. One can object that they may be eliminated and turn it over to computer. But this is as bare as the affirmation that proofs may be formalized. Four colorations mishap gives a good voice for our position.

Unfortunately, we do not know anybody who should hae explicitly stated this enigma, what a strict proof is? Nevertheless, we believe, people doing with mathematics would readily accede that formal proof is a necessary thing, but it is needed rarely.

"Strict" and "formal" proofs are usually mixed up. That's wrong. Besides, the following state of the problem is appropriate: the absence of the discernible idea of strictness may lead to a lot of undesirable questions, what may a reader be like, for example.

Book [3], already cited, has an explicit remark that proof should depend upon the reader it is addressed. "Euclidean demonstration of the infinity of simple numbers set may be compressed into one string of text for a professional". Refuse of a set of means of demonstration also takes place - it was considered above that a strict proof should go without figures (an example, what can this limitation provoke is given in [3]). Formal parts of a text are supposed to have the major importance, that is achieved to the detriment of non-formal elucidations and, consequently, distinctness of the fundamentals. In introduction to [8] V. Arnold points out (in attempt to withstand this temptation): "The author tries to avoid axiomatical deductive style, characterized by unmotivated definitions disguising the fundamental ideas and methods; the latter is explained tete-a-tete like parables".

After all, overegard of the necessary extent of text formalization takes place. Consequently, great efforts to give shape to a mathematical work are spent for "coding" that entails the adequate expenses to "decode" while reading. This is especially typical for combinatorial deductions (see the example hereafter).

The concept of strictness itself is somewhat subjective for it depends upon addressee. The higher are the capacities and the comparative level of the latter, the greater abyss lies between strictness and formalism. That's why proof exposition may be less formal in more simple places that alleviates perception. Appropriately the comparisson with the quality of photos about difficult water or mountain compaign: there are a large number of photos during the rest and very little about the difficulties of undergoing the threshoulds or glaciers.

In terms of the foregoing, one should not surprise, why physicists, engineers and non-mathematical school teachers, in other words those who do with mathematics not at the professional level, are often addicted to formalism. Moreover, many of the mature mathematicians have got through a particular stage of their career, when formalism happened to be an absolute self-contained ideal for them. Unfortunately, this is very typical for modern teachers of mathematics and senior school students in mathematical colleges. We shall consider it in more details later.

As long as strictness immediately depends only upon reader's level, the way to strictness swerves from one to formalism. Strictness may be achieved just by delivering ideas and recipes to create a formal proof anyway. The larger is the circle of readers, the more strict is the text, not more formal. The more advantageous is the demonstration of ideas, the wider range of opportunities has the reader to interpret it, the greater is the difference between strictness level and degree of formalism, the less rigorous are the demands of formal accuracy, the shorter is the distance from exposition to human reasoning. This, in turn, eases apprehension thus giving more effective representation of the particular ideas and increasing the space between strict and formal things, and after all total revision of the very rules of formalization becomes possible, for the only reason that is our non-formal cogitating.

It is worth mentioning that formal proof concept is in some extent conditioned by background. Mathematical analysis development witnesses principal impossibility of doing without intuition, just because one cannot establish congruity of mathematics in general [4]. The act of confidence in validity, usefulness, interest in the result obtained is the ground for scientific knowledge. What would be implied in the statement such as "basing upon basing necessity" or "making certain in practice that practice is the criterion of truth"?

If formalized (like put into a model), some facts may be mislaid. That's why formalization is often practicable to be fulfilled in different ways, like many models of a single phenomenon may be laboured out. The different definitions of infinitely small values and differentials in ordinary courses of mathematical analysis and non-standard analysis corroborate this [9–10].

Formalization itself subsumes an issue: it is sufficient to deliver a formula for roots to prove resolvability of the corresponding radical equation, but demonstration of non-resolvability requires formalization of the very idea of resolvability. In the same way, the formal definition of proof appeared after D. Gilbert had set forth the problem of possibility of verification truth of any mathematical statement. After that K. Gödel succeeded in non-resolvability of this question manifestation. The motive may be somewhat as follows: the formalized idea is investigated in limits of mathematics itself (the latter, as noticed, has true, but unprovable statements). This is also the case when the procedure of resolution is compiled by computer. To work out this issue we usually imagine an ideal devise (one of Turing or of Post) whom we shall explain what to do, i.e., formalize. The level of formalization depends upon our facility to do without the devise. Whether or not formalization is possible, is the matter of outstanding significance (like the unambiguous definition of "etc." is impossible owing to non-standard models existence).

So, the way of exposition, maximally adequate to human cerebration (in particular question), primordially blurred, different from formalism, is the criterion of strictness. Hence, there is no need for exposition of all the steps of the proof as the linear text. Because people are generally cogitating by images. J. Adamar refers to G. Birkhoff as a person possessing a very rare kind of thought, “cogitating by images” [11]. So, graphics and other options that have become available with the advent of computers, may be utilized to mould proofs, with the text comments being settled down in non-linear way.

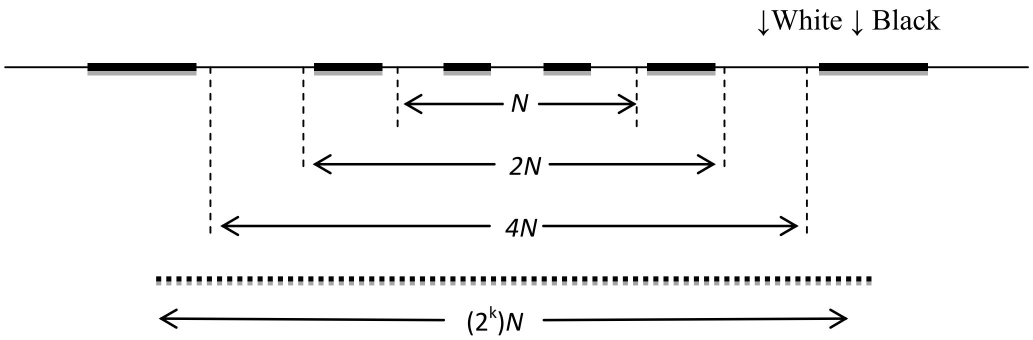
Finally, we have approached the “mathematical commix” (m-comix) idea. This is something made of any attainable material, such as primary text and graphical images, designed to carry out a formal proof. Simply speaking, m-comics is the summary of the formal proof or more vividly – it is the colored sheet of paper with the help of which the reader can carry out the strict proof of corresponding theorem. In this connection we cannot but recall the history of creation of [12] book illustrations that, by the author A.T. Fomenko, once served as the conundrum of homotopia topology lectures.

Below given, is the example of one combinatorial theorem demonstration by means of m-comix. These are the combinatorial ideas that make up the extreme difficulties for understanding, if delivered in the traditional way. The idea how the proof was created, always seem vogue.

THEOREM (Van der Vaerden). Suppose *k* and *l* are natural numbers. Then exists such a natural *n*, $n = n(k, l)$, that any coloration of *k* colours of any segment of the natural set of length *n*, contains a single-colored arithmetical progression of length *l*.

NOTE. In case of coloration of *k* colours of the whole natural series, there exists an arithmetical progression of one colour of any finite length, but one of the infinite length may be not available.

Example: both white and black segments of any length are available.



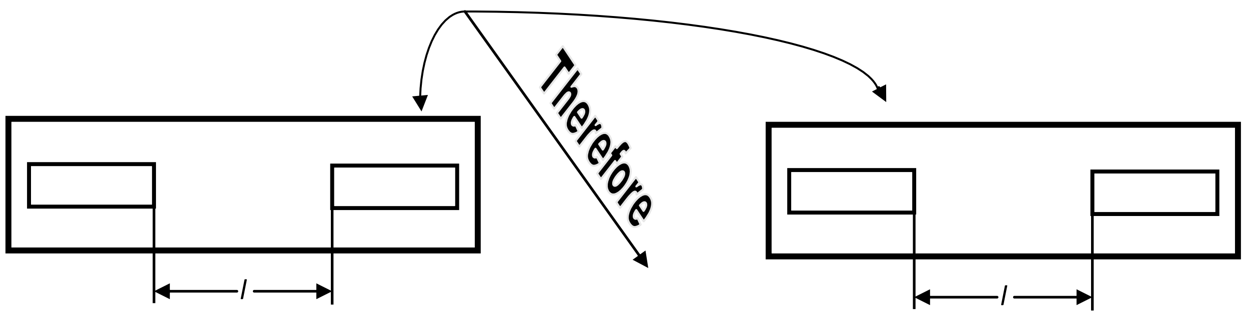

any infinite arithmetical progression contains already white and also black number

Here after the stripe is the image of natural series and quadrilaterals are images of segments of natural series.

**The search of the progression with the length equals to 3**
**( ‘k’ – is the number of colours):**

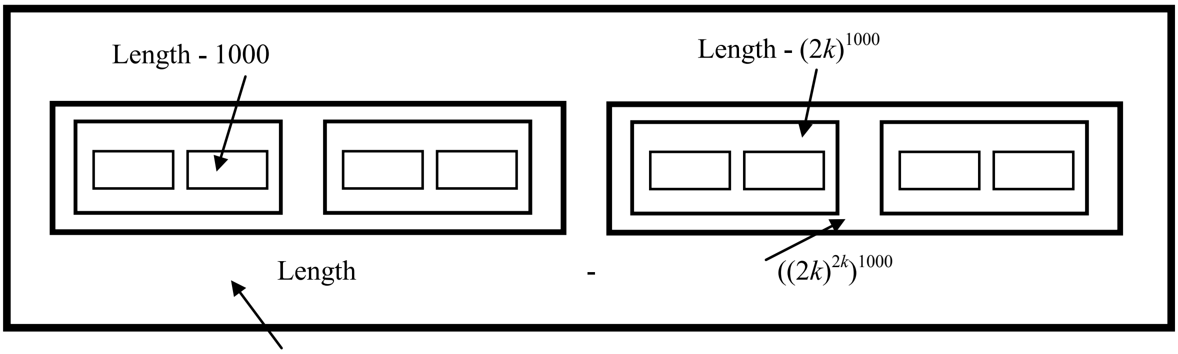
In the piece with the length $(3k)^{(3k)1000}$ one can find 2 identical pieces with the length $k^{1000}$

In the piece with the length $(3k)^{1000}$ one can find 2 identical pieces with the length 1000



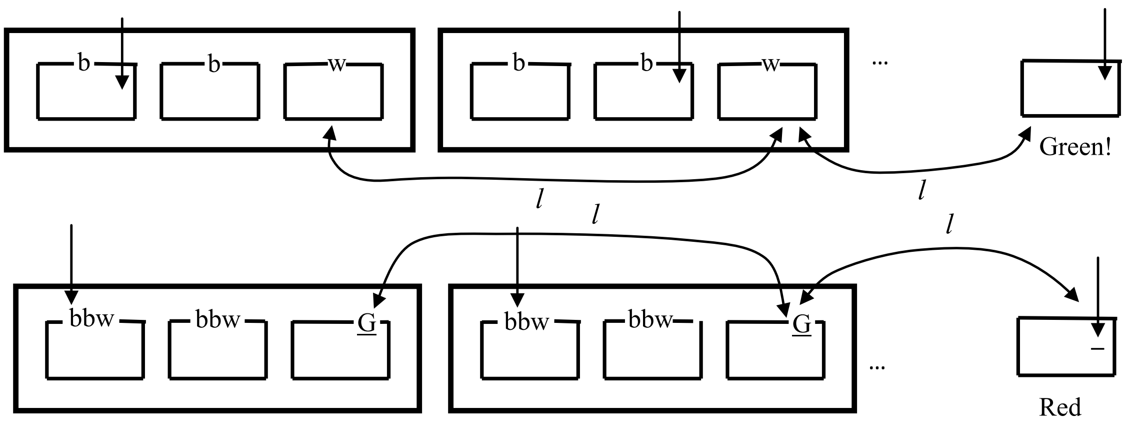

4 pieces of the same length 1000 and distances between them are the same also.

**Similarly, we can build "three-story sandwich" also**



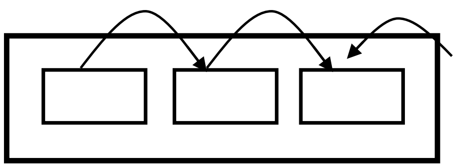

It's length will be $(((2k)^{2k})^{2k})^{1000}$ ... and so on: *N*-storey "sandwich", with fixed *N*.

**How to build a progression of length 3**




Generally, if we have *k* colors, then "*k*-storey sandwich" is needed.

Clarification: symmetric piece of the next stage

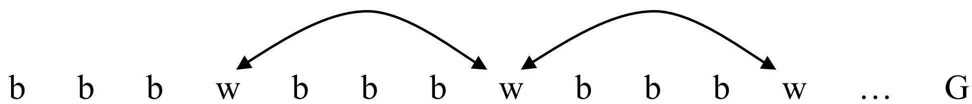


it is necessary to require that trapped inside a piece in their hierarchy.

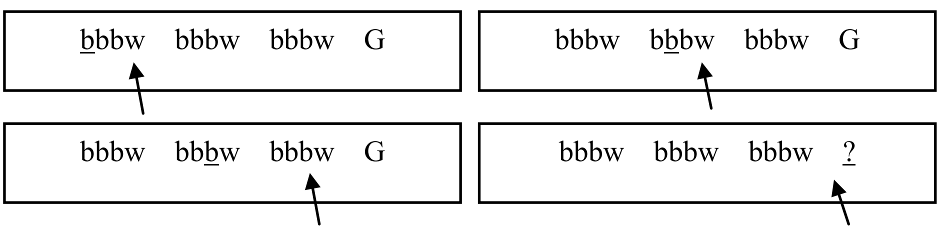
Therefore, the first floor length must be taken not as $2*k^{1000}$, but $4*k^{1000}$.

**Case with progression of length 4**

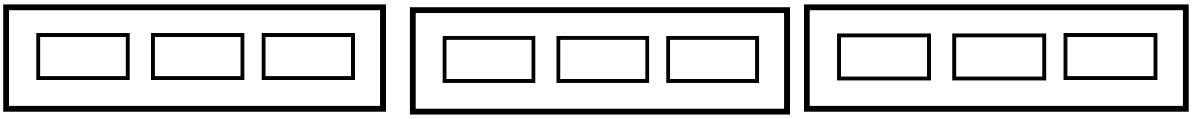
Blue dream:



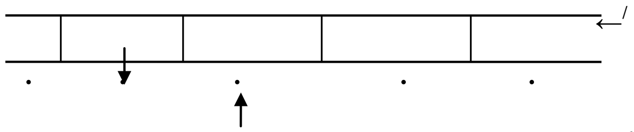

Fork:



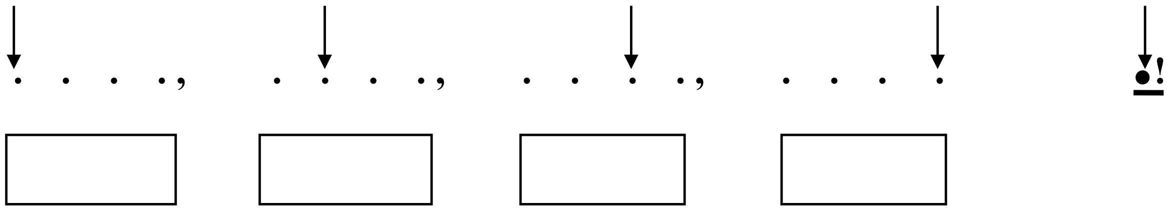
“Sandwich” is obtained similarly:



Implementation: encode pieces of length $l$ by chips. |

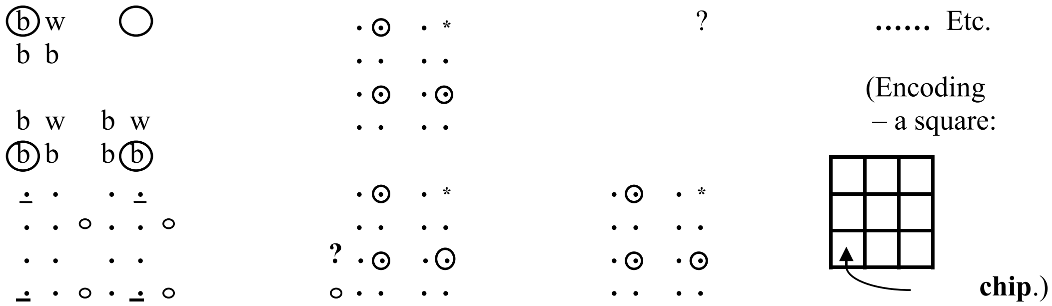



Colorized chip – painted piece. Chips are colored in $l^l$ ways.
Then we find the progression of length 3 if a piece is fairly large.



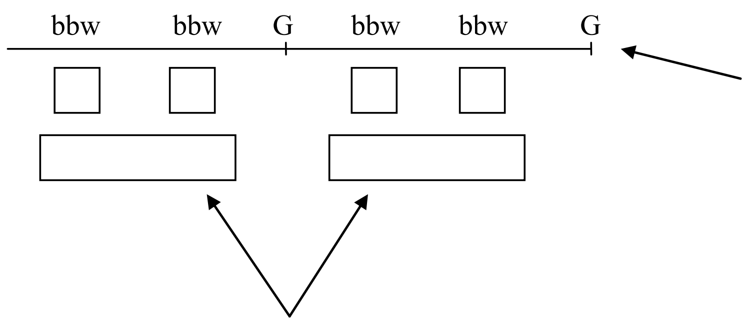

**Flat and spatial generalization**

There are 4 pieces of the same color in the vertices of a square. Visualization:




Assessment is carried out as before.

The foregoing m-comix comprises the entire proof of Van der Vaerden theorem, generalized over the case of any dimension of Euclidian space [23], although corresponding multidimensional theorem hasn’t just been worded! We may demonstrate how the esteem attained may be reinforced:




It’s enough.
Let’s call this as ‘type’.

Not necessarily,
to all repeated.

How many types of depth $l$?
(Figure realized 3 as a depth)
The number of colorings – $l^l$.
But besides that there are ‘steps’ – distances between the nearest black, white, blue ...

So, if still $N(k, l) = B(k, k, l{-}1)$, than the assessment for $l = 3$ may be improved as:

$$B(t+1, k, 3) = \underset{\substack{\text{number}\\\text{of modes}\\\text{for the}\\\text{greatest}\\\text{step}}}{k^{(t+1)}} \times \underset{\substack{\text{number}\\\text{of modes}\\\text{for the}\\\text{second}\\\text{value step}}}{B(t, k, 3)} \times B(t-1, k, 3) \times \ldots$$

So, $B(t + 1), k, l) = k^{(t+1)} \times B(t, k, l) \times B(t - 1, k, l) \times \ldots$ .

It took just one lesson to demonstrate the proof of this theorem, supplied with all the additional comments, to seventh-form pupils of Moscow Mathematical Society School in terms of the foregoing m-comix. As for the traditional proof, its single-dimensional application, deprived of reinforced evaluation of $n(k, l)$ delivered in [13], takes 12 pages of abstruse deductions. The author A.Ya. Hinchin forewarns the reader in the preface to this book that it may take more than a week to get at it. Certainly, not awkwardness of the author in demonstration is the reason for it: his numerous books and listeners maintain the converse; the general failure of a customary exposition to explain rather subtle combinatorial ideas takes place.

Traditional proof is held by means of induction in $l$, the value $n(k, l)$ is introduced, the recurrent sequence is defined:

$q(0) = 1$; $m(0) = n(k, l)$; $q(s) = 2 \times q(s - 1) \times \times m(s - 1)$; $m(s) = n(k^{q(s)}, l)$.

Then the statement that $q(k)$ may be taken instead of $n(k, l{+}1)$ is proved. Geometrical interpretation of $q(s)$ is the length of s-stored "sandwiches" introduced in m-comix, sense of deuce is mutual non-overlaping of the progressions, but the reader must still be unaware of this. Then "strict" notion of the sections of the similar type, i.e. coinciding under superposition, is introduced. The case of collective hierarchy is considered immediately, and the forks are strictly proved by means of introducing a large number of indexes.

The outlined way of proving is not convenient for the three ideas are simultaneously involved in: fork, multistory sandwich, and coding. This results to bad understanding, for deductions, involving a number of not similar ideas at the same time, are difficult to get at. Successive introduction of the necessary ideas and review of particularities enormously swell the text. M-comix gives the reader a chance to go through formal aspects of the matter oneself and succeeds in elimination of these drawbacks.

Not only abstract motivation, such as "purquoi pas?" makes the concept of m-comix vital. Neither does the wish to drive exposition of the particular theorem proofs, considered difficult for understanding before, to paragon. The matter is that one can have at least two different views of mathematics: "administrative", i.e., characterized by consideration of some of its sections, and "industrial" that comes to study of the fundamental ideas, and their reflection in different situations in different branches of mathematics.

The administrative view is used in all the books referred, instead of, probably, [14] and other books of its author. The industrial view is present generally in papers of methodologists that are usually out of the attention of the professional mathematicians. We believe, every substantially new conception is always important, and methodologists' addiction to "industrial" approach is worthy of the serious attention. By the way, there are not only mathematicians among those who have this sin, it's enough to recall the relation of biologists to Villis and Jule's law in systematics [15], that was extremely negative at first, for this law might be also applied to other sciences, such as sociology, for example.

Coming back to mathematics, we must say that the question of relative importance of the both approaches cannot still be answered exactly for the reason of immaturity of the latter. But some papers written by specialists in applied mathematics, who practised the "industrial" approach have come out at late. Indeed, if one reviews the sections of mathematics, commonly used in other sciences, such as analysis, linear algebra, probability and statistics, it may be inferred that mathematics is commonly implemented as the course of ideas, not formal recipes. This is especially the case with arithmetics that had become the part of the human being and culture long before.

If the purpose is studying ideas, exposition should be desirably close to their primary view when they were engendered. As the rule, human cerebration is pictorial [11]. Computer visualization and polygraphic means, accurately involved, could become the best realization of m-comix concept.

Since the industrial point of view is important for investigations in the field of artificial intelligence, m-comixes could be helpful to elaborate this research. In this connection, it's worth mentioning that m-comixes would promote the creation of the algorithms, designed for parallel calculations, like ordinary texts are suitable for single-tasking. Pictorial cerebration and usefulness of m-comixes somewhat confirm the view of human brains as parallel processing system.

There are many objections against the idea of m-comixes. Observed below, are the principal ones.

The first is basing upon the necessity of texts unification, i.e. making them easily understandable for the wide range of readers. This is the cultural role of standards. This is the eternal conservatism of culture, interfering with any breaking the style, that is especially evident for written texts (we realize that coding text results in self-violence, alas, this cannot be evaded. But the text, written in ecstasy, is enchanting and the reader takes it in the appropriate way. If the text is resistant to lay in the Procrustean bed of standards, the author commits violation over the text, oneself, and the reader), when there is no feedback with the addressee. The author often commits violation over the reader and himself for "duty feeling", in spite of the conditional essence of cultural standards!

Indeed, m-comix may be unreadable for the reason of its ambiguity, if the above fact gets away from consideration. This ambiguity emphasizes the ability of human mind to get solutions of a plenty of tasks, in this or that way associated with the principal one, due to life experience, simultaneously. A single picture in front of the cerebral look, a single m-comix may comprise solutions of many problems, like a single idea is efficient for seemingly different applications.

There is no surprise. The model of the peculiar phenomenon, illustrated by m-comix, is worked out, disregarding everything non-essential, and consequently may correspond to a number of phenomena. The bare idea, usually trivial without the tinsel, is the model of the phenomenon, refined to the extent of triviality, but this simplicity is essential for resolution of very complex problems. Disciple principle is a good example in this connection. This famous idea seems natural and then looks obvious, but it may be not distinct in a complicated task. We shall answer this objection in the following way.

First, there is the sufficient number of addressees, who are able to get at the essence of the proof demonstrated in terms of m-comix, that gives justification of this approach.

Second, comments may be attached to the picture, that shall ensure unambiguity of the idea.

Third, as it is often the case for the traditional texts of proofs, m-comix may comprise one sense, and the deciphered text – the other. This effect may appear if the author and the reader are not coeval or speak different languages. The same situation would happen in astronomy and chronology when astrological texts were handled, and in chemistry when it dealt with alchemy. Modern interpretation of mathematical results of Ancient World and Middle Ages such as magic squares and irrational numbers, of course, goes without mysticism and occultism. By the translator of [16] book, that is closer to nowadays, it has certain faults and gaps – this is the effect of change in terminology of the theory of functions of a complex variable that came about for the last forty years.

We may come across the matter of the same kind in any science. Investigating a complicated phenomenon, we usually make concessions to nature and do with the problem, similar to one in consideration, that enables creation of mathematical model. The investigator inspires to follow the principle of involvement in the model of all the factors that influence it in approximately equal orders.

The other objection lies within the obvious intuitive grounds of m-comix, that may sometimes mislead. The numerous illustrations are given in [7], such as curves of non-zero area, domains, devoid of area, etc.

We have partially answered this objection discussing the concept of strictness. This is not universal power of intuition the matter, m-comix shall deliver a proof preparation, and the reader shall have enough potential to restore the latter. The permission for explicit use of intuition, followed by "responsibility", is the matter. The omnipresent fundamental philosophy principle of verification is hereby reflected. No matter, if a physicist has adequate means to observe a quantity of one's interest at the disposal, the possibility itself is significant. In logics finiteness is determined only by principal contingency to verify truth of the statement for a limited number of steps of computer, whatever it is. Quite the same, strictness is the concept of formal possibility to verify.

The third objection is the relative equivocation of m-comix creation. The answer is obvious: creation of m-comix is a complicated task. But first, traditional texts of proofs are the particular examples of m-comixes, as long as utilization of modern computers is voluntary. They often facilitate getting through the proof details, as the creation of m-comix elucidates the ideas animating the proof. Second, this problem is not the case, when textbooks are designed, for any immense efforts are justified in this case. We have still been discussing strictness as a possibility to give formal interpretation, but now we are able to generalize this concept. Human cerebration is not crystal-clear, there are different stages of legibility of exposition (and appropriate cultural standards), that form the entire hierarchy. So far, the legibility rate of mathematician is lower than one of computer, physisist's one precedes mathematician's, and humanist's – engineer's (more circumstantial division is possible). Let's define cerebral strictness level as the highest attainable rate of distinctness the idea may be lead to, by neatly methodological efforts of the reader, or it may be lead to, if considered from philosophical position.

Nebulous reasoning is more powerful, it yields the result, probably erroneous, post-hastily. Modern computers possess the absolutely distinct reasoning, and their "creative potential" is famous. In the other hand, the same idea, delivered at a higher level of distinctness is more valuable. The result may be obtained in an easier way at a low distinctness level, and then elaboration of legibility, i.e., definition, establishing, and ,after all, exposition in oral form (lecture or report), and in written form follow.

Now we'd like to pay attention to the demonstration of oral proofs. We are sure, that m-comix idea has initially been contained by them (certainly, only good-level lectures are taken into account). These are ideas and methods that animate proof, emphasized firstly. Lectures, overfilled with extra details such as long mathematical operations are inevitably dull.

Such a step aside from paradise of formalism delineated itself at the beginning of the XX century. Conversely, quality of oral proof had been estimated from the point of its formal benefits, i.e. readiness of its immediate written interpretation, before. For example, book [17] represents the notes of lectures on probability delivered by A. Lebedev, almost unprocessed, made by one of his listeners, M. Lyapunov.

Thus far we come to the question of the role and the facilities of m-comix perception and its practical realization by means of modern computers in mathematical results demonstration to the audience. In our opinion, lectorial variant of m-comixes, by the way, more familiar than its written analogue, comes to schemes of proofs. Of course, a good lecture dedicated to demonstration of a particular theorem, is rather m-comix than a formal proof. Ancient legend tells that when a geometry teacher was asked by his pupil, why a geometry fact is true, he drew a pattern and ordered: "Look!".

It's necessary to point out that only tutoring lectures are in question here. Additional visual information, if available, delivered by placards, or obtained from the screen, simplifies and accelerates the proof, but professional mathematicians are known to prefer looking at the blackboard but not at the screen during reports, regulating by this the speed of delivery. This example portrays the difference between pedagogic and pure mathematical creation.

The main demands to schemes of proofs, given by lectures, are as follows. They must be well structured; different stages of deductions are required not to come across each other - even less complicated proofs of auxiliary statements. A scheme is good if approximately equal efforts for getting at every stage are needed.

Negative consequences of the lack of exposition experience, a reporter may probably have, are mentioned in introduction to [18]: Moscow State University mathematical circles failures that took place in 1936–1937 were generally caused by the style, dominating at that time, which consisted in material exposition by pupils themselves – they could be thoroughly aware of the matter at best, but had neither tuition experience, nor necessary erudition.

We shall give an example of the scheme of proof of Euler formula for polyhedra, this theorem belongs to the golden fund of popular mathematics. Unfortunately, it is out of the school course of geometry, although the graph theory, covering this theorem in fact, has become as important, as the signs of triangles coherence are, for example.

***Theorem.*** The following relation is true for any simple polyhedrum, (i.e. one, appearing fram a ball by cutting off segments or their parts resulted from a previous cut off): $V - E + F = 2$, where V – number of vertexes, E – number of edges, F – number of faces.

***Comments.*** This theorem is usually delivered to pupils of the 6th or 7th form of Moscow Mathematical Society School, and takes two academic hours. Use of stereometry theorems is not permitted for such an audience. Nevertheless, Euler result (Decart – Euler result) may be obtained just in terms of elementary plane graph theory in the generalized form, regarding probable non-connection of the graph.

***Proof scheme***

1. Project the surface of the polyhedrum to the embracing sphere, centred inside the polyhedrum.

*Questions*

a) What kind of lines will the images of the edges of the polyhedrum be?

b) In what way will it influence the sum $V - E + F$?

2. Stereographically project the sphere and the grid of arcs, obtained on its surface, to the plane that has no interception with the sphere.

*Questions are the same as at* (1)

3. Generalize the problem: introduce the concept of graph with arbitrary edges, if it wasn't given previously. State a question about the sum.

4. Simplify the graph: locate all the vertexes of degree 1, having previously introduced the definition of degree, if not known (we shall further suppose that all the notions are already familiar to

the audience, otherwise acquaintance with any shall immediately precede its utilization by default), remove all of them and the edges adjacent to them, repeat until it is unrealisable. Trail for the change of the sum in question.

5. Simplify the graph: locate all the vertexes of degree 2, remove them, changing a pair of corresponding edges for single edges, until it is unrealisable. Trail for the change of the sum in question.

6. Simplify the graph: locate all the loops and multiple edges, remove the former and change the latter for single edges, trail for the sum in question.

7. Repeat procedures 4–6, if attainable.

8. Simplify the graph: change any edge and abutting vertexes for a single vertex, connected with all the vertexes, the two removed vertexes were connected with, and only with them, repeat until it is possible. Trail for the change of the sum V – E + F.

9. Proceed p. 6. Come to the edgeless graph. Calculate the sum V – E + F. Introduce the concept of connected compound – the latter is a pre-image of any vertex of the finally obtained graph.

Question: to how many vertexes will the initial graph, relative to vertexes and edges of a simple polytheism, be transformed?

10. Generalize the formula obtained over the case of any polyhedrum in terms of transition to toroidal graphs, if time and attention of the audience enable.

***Comments on the scheme of proof***

1. Brief immediate comments are given during the text itself.

2. Diversion to more detailed survey of stereographic projection is permissible, if the level of the audience enables.

3. One should ignore some ambiguity of the notion of "arbitrary" line. The proof, lying within the above scheme, may be processed in case of linear edges representation, but this immensely snowballs it. Besides, the theorem stating the possibility of putting every planar graph into a plane graph is usually given at the next lessons [20]. This is the reason why the concept of plane graph is postponed, the more so as the immediate application of the theorem follows by verification that some simplest graphs – full one of 5 vertexes and full dicotyledonous of three pairs of vertexes – are not planar.

4–6. These boils down are not inevitable in sense of pure logics – simplification of p. 8 and loops deletion, conserving the investigated sum, is sufficient. But the experience of exposition in accordance with the scheme shows that they apparently facilitate getting at the matter of p. 8, for this transformation is the most complicated in the proof. In addition, this section of the scheme delivers a consistent example of algorithm that will be helpful for future study of mathematics.

10. This section is not necessary, of course.

Meting lectures out, restricted to schemes of proofs, gives a teacher a brilliant additional opportunity of non-formal quiz giving, for he could not only call on the pupils to retell the previously administered material, but to answer the particular questions covering details of this or that part of the proof.

In our opinion, a set of themes (lessons, fragments) contained by textbooks, may be constructed by pupils themselves. this could preserve the inner manner, logics, and technique of meting out material, that is closer to them and could be better interpreted. A teacher who gets about processing material, presented by pupils, must scrupulously do with style and logics of exposition to preserve it unchangeable, much more accurately, than an editor of a scientific or artistic journal, for the former is designed for children.

There are, certainly, some drawbacks, characteristic of this approach. One of the principal is a priori a problem of mutual attachment of some particular brightly elucidated themes, represented in the way, diverging from the traditional standard, and their unification into the whole text. This is the kind of problem that, certainly, cannot be solved by only pupils' means.

Rivalries between the lectorial proofs from courses, delivered at different schools (institutes), may be extremely valuable as a statistical experiment. Different types of proofs of the same theorems are expected to be demonstrated. Annual or half-a-year results of work should be fixed and compared in accordance with different criteria (percentage of understanding, presence of commonly spread mistakes, etc.). Those proofs which win these competitions and meet the requirements aforesaid, could be the best foundation for textbooks creation. Such a collective way of textbook combination could eliminate at least some of the typical incongruities of the existing courses, for even the best of them are not equivalent in their parts. Their author – adept in a specific group of the delivered topics – can have one's own beloved themes, which can be exposed by him better than by others. In addition, borrowings almost inevitably arise in the chapters that demonstrate classical results for entire coherence.

**Conclusions for tuition**

1. Visual image of proof process, provided by m-comix, generally gives an answer the question

positioned at the title of book [14] – how to solve a problem?, – whose author demonstrated mathematics in statu nascendi for the first time. This hadn't been displayed before neither to teachers, nor to audience. D. Poya states there that the both aspects of mathematics – inductive (experimental), to which the book is devoted, and deductive (Euclidean) – are as ancient as mathematics itself. It's worth mentioning that this conclusion is just for any science along with mathematics and investigation of result substantiating dynamics is as important for mathematics, as the analysis of process of its obtaining.

Transition to use of m-comixes in proofs is able not only to give fake of independent activity, that often takes place in pupils' dialogue with computer - "science is the thing that may be made by use of computer, except for games, in addition to obligatory program" – but also to apply their penchant to computers to problems, still not resolved. [21] gives the examples of problems whose formulation goes without special terms and shows the benefit of use of computer evident. The problem of "hailstone numbers" is the most famous: if the recursive action as follows result in 1 for every natural N: if N is an even number, do with N/2, otherwise do with 3N+1.

2. An interested reader should better deal with the explicit system of references to auxiliary intermediate statements, consistently used and acquainted or evident, or with familiar proofs, than with the formal texts, to understand the things made by the authors of a particular work. This type of proof text structure, reminiscent of the scheme of correct waypointers, used in library trees and databases, facilitates considerably more explicit representation of the inner dynamics of proof process. This dynamics should be shown and utilized as soon as it is possible in educational system (at primary school), with its use automatically giving a chance to classify pupils on the base of the speed of getting through the tree and depth achieved.

Appearance of such books as [22] may be interpreted as an achievement in the foregoing direction. Along with positioning problems and detailed resolution there are chapters of "first-second directions" and "third directions" which prompt the possible steps of problem resolution to the reader, interested in independent research. Putting computers into play will promote the further introduction of larger scale exposition dynamism, i.e. in limits of one lecture, class, theme. Notes of material, delivered in a proper way (giving the opportunity of reproduction of exposition dynamics), will be the natural generalization of the ground signals – one of the major elements in terms of Shatalov method.

3. It's important to realize the evolution of pupils' attitude to the concept of strictness. None of them really gets at the sense of the concept of proof at the very beginning of study, although many of them are able to make use of some oral models of argumentation, such as logical connections, like "in case of", "therefore", etc.

Then they apprehend the idea, when the statement may be referred to as proved. Later, visual accuracy and formalism of argumentation often becomes a self-contained purpose, and a pupil (if he goes in for mathematics) feels fastidious about minutenesses. At this time comprehension of what shall a strict proof be like may come. A teacher shall take this chance and begin processing proofs of negative affirmations, especially proofs of impossibility, that require formalization of notions, seeming quite obvious. From this point of view, constructions with a single ruler are valuable, especially the problem about impossibility of drawing a perpendicular to a given line by the foregoing means. Optional course of logics and the theory of the sets could be also helpful.

To give shape to the concept of strictness, m-comixes may be used. They also can give pupils the opportunity of passing through proofs of the most complicated and profound theorems themselves and simultaneously prepare them to get at the fact that m-comix is enough strict proof, if it may be formally interpreted by means of addressee, i.e. a pupil oneself. The numerous questions of solution reconstruction from the main idea, given by textbook (unfortunately, not by every one) in the section, named "Directions", tasks to write down a proof in a traditional way from its m-comix, problems of proper getting up a solution (not calligraphy is implied, of course) are characteristic of this stage. Alas, no one teaches to get up properly: simple and typical tasks have the same design, but it is impossible to require the class to solve too complex problems. Professional mathematicians also come across the problems, really difficult for them, rarely, and therefore usually attain designing skills rather late – the first paper is often returned by editorial stuff for completion.

Problems of solution renewal from given m-comix may be beneficial in this connection. Appropriate, is the analogy with young poets and artists, who often write plagiasm while study. This sort of exercise cannot be the source of new results itself, of course.

No doubt, difference between formalism and strictness should be illustrated to a pupil interested in mathematics, but one should avoid direct notation: this fondness of formalism is based on the need for mathematical reasoning standards apprehension and getting skills of drawing thoughts to the end, and a pupil shall get tired of these games. There is the common rule for the teacher to follow: the more problems are solved by the pupil oneself, the better.

After all, “fatigued of formal operations plenitude”, the pupil shall come to the idea that “mathematics is language” (Gibbs). To enjoy substantiating, one shall be able to speak the proper language and not to be verbose: the iterlocutor (teacher or listener) can say much about tasks and theorems oneself.

The next step clarifies that the statement is proving, if the addressee is able to formalize it. Similarly, mathematical definition is “the proof of existence” of a formal object, appropriate to its prototype intuitive comprehension. But this stage, if comes, goes after school.

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